Pergamon

J. Appl. Maths Mechs, Vol. 58, No. 2, pp. 207–221, 1994
Copyright © 1994 Elsevier Science Ltd
Printed in Great Britain. All rights reserved
0021–8928/94 $24.00+0.00

0021–8928(94)E0044-B

APPROXIMATION SCHEMES FOR CONSTRUCTING MINIMAX SOLUTIONS OF HAMILTON–JACOBI EQUATIONS†

A. M. TARAS'YEV

Ekaterinburg

(*Received* 25 *February* 1993)

A grid algorithm is proposed for constructing the optimal guaranteed result function (which need not be differentiable) in control problems. Wherever it is differentiable, this function satisfies the Isaacs–Bellman equation, which is a first-order partial differential equation of Hamilton–Jacobi type. A convergent finite-difference method is proposed for Hamilton–Jacobi equations. Unlike the classical grid method, in which one approximates the gradients of the unknown function, which need not necessarily exist, this method requires the computation of subdifferentials of locally convex hulls. Underlying the method is the concept of a generalized minimax (viscosity) solution [1–4] of the Hamilton–Jacobi equation, with the corresponding infinitesimal constructions—directional differentials and subdifferentials—replacing the classical derivative.

Generalized solutions of Hamilton–Jacobi equations and their numerical computation attracted considerable attention at the beginning of the 1960s [5–10]. When a general theory of generalized solutions emerged in the early 1980s [1–4], attention again turned to numerical methods. Various approximation operators (AOs) were examined [4, 11–15] within the context of the theory of viscosity solutions [3]. Various workers have proposed a general scheme for proving the convergence of approximation schemes (ASs) and for obtaining convergence estimates [4, 13]. Explicit schemes with AOs of the Lax–Friedrichs type have also been considered [4], and an implicit AS has been investigated in detail [13]. Another topic investigated is "essentially non-oscillatory" ASs with operators of Godunov and Lax–Friedrichs type, using local approximations of higher than first order (where the latter corresponds to piecewise linear approximation) [12]. In the context of Godunov's scheme, consideration has been given to "maxmin" and "minmax" operators, and cases have been discovered in which these relations yield an exact formula for the viscosity solution of the Riemann problem.

The AOs proposed in this paper differ from the algorithms mentioned. Obtained in the theory of optimal guaranteed control (differential games) developed by Krasovskii and his school, these operators are based on the results of [14–19]. Underlying the operators are notions from convex and non-smooth analysis [20–22]: locally convex hulls of the function being approximated and subdifferentials of such hulls. The AOs may also be obtained by double constructions in terms of subdifferentials of locally concave hulls. These notions are connected in a natural relation to Dem'yanov's subdifferentials and superdifferentials [20], which may be used to obtain a universal AO.

It will be shown below that our AOs satisfy certain sufficient conditions [13] for the convergence of the corresponding ASs. However, they differ from the operators of [4, 11–13]

†*Prikl. Mat. Mekh.* Vol. 58, No. 2, pp. 22–36, 1994.

in one fundamental respect: they may be defined outside the elementary "rhombus" of phase space. Therefore, they do not rigidly connect the time-approximation steps with the approximation steps in phase space. In what follows we shall also compare our AOs with those of [4, 11–13].

1. THE OPTIMAL GUARANTEED RESULT FUNCTION

We consider a control system whose dynamics over a time interval $T=[t_0, \vartheta]$ is described by a vector differential equation

$$\dot{x}=f(t,x,u,\upsilon)=h(t,x)+B(t,x)u+C(t,x)\upsilon \tag{1.1}$$

$$x\in R^n, \quad u\in P\subset R^p, \quad \upsilon\in Q\subset R^q$$

where x is the n-dimensional phase vector of the system, u is the control and υ is the noise vector. The sets P and Q are convex and compact. The function $f(t, x, u, \upsilon)$ on the right-hand side of system (1.1) is assumed to satisfy the following conditions.

(f1) Joint continuity in all variables.

(f2) The Lipschitz condition with respect to the variable x

$$\|f(t,x_1,u,\upsilon)-f(t,x_2,u,\upsilon)\|\le L_1(D)\|x_1-x_2\|$$

for all $(t, x_1)\in D$, $(t,x_2)\in D$, $u\in P$, $\upsilon\in Q$.

(f3) Continuability of solutions: a constant $\kappa>0$ exists such that

$$\|f(t,x,u,\upsilon)\|\le \kappa(1+\|x\|)$$

for all $(t,x,u,\upsilon)\in T\times R^n\times P\times Q$.

(f4) The Lipschitz condition with respect to the variable t

$$\|f(t_1,x,u,\upsilon)-f(t_2,x,u,\upsilon)\|\le L_2(D)\|t_1-t_2\|$$

for all $(t_1, x)\in D$, $(t_2, x)\in D$, $u\in P$, $\upsilon\in Q$, where D is a compact set, $D\subset T\times R^n$.

The control problem considered here is to guarantee minimization of the functional

$$\gamma((x(\bullet))=\sigma(x(\vartheta)) \tag{1.2}$$

on the trajectories $x(\cdot)$ of system (1.1), i.e. to find a positional control $U^\circ=U^\circ(t, x)$ that will provide an external minimum in the relation

$$w(t_*,x_*)=\min_U \max_{x(\bullet)\in X(t_*,x_*,U)} \sigma(x(\vartheta)) \tag{1.3}$$

and accordingly to determine the number $w(t_*, x_*)$, called the optimal guaranteed result. Here $X(t_*, x_*, U)$ is the set of trajectories of system (1.1) that leave the initial position (t_*, x_*) and correspond to the positional control $U=U(t, x)$ [15].

It is assumed that the function $\sigma: R^n\to R$ in the functional (1.2) is Lipschitz continuous

$$|\sigma(x_1)-\sigma(x_2)|\le L_3(D)\|x_1-x_2\|$$

for all $x_1\in D$, $x_2\in D$, where D is a compact set $D\subset R^n$.

The function that assigns to each position $(t_*, x_*)\in T\times R^n$ an optimal guaranteed result

$w(t_*, x_*)$ is called the optimal guaranteed result function or utility function.

It is well known that the utility function $w(t, x)$ is Lipschitz continuous and is consequently differentiable almost everywhere; at its points of differentiability it satisfies the Isaacs–Bellman equation, which is a Hamilton–Jacobi type equation

$$\partial w / \partial t + H(t, x, \partial w / \partial x) = 0 \tag{1.4}$$

It also satisfies the boundary condition

$$w(\vartheta, x) = \sigma(x) \tag{1.5}$$

where

$$H(t,x,s) = \langle s, h(t,x)\rangle + \min_{u \in P}\langle s, B(t,x)u\rangle + \max_{u \in Q}\langle s, C(t,x)\upsilon\rangle \tag{1.6}$$

is the Hamiltonian of the control system (1.1).

Solutions of problem (1.4), (1.5) have been rigorously defined, and existence and uniqueness theorems have been proved in the theory of minimax (viscosity) solutions of Hamilton–Jacobi equations [1–4, 16]. One of these definitions [16] follows.

Definition 1. A Lipschitz continuous function $w(t,x)$ is called a minimax solution of problem (1.4), (1.5) if the following inequalities hold for all $(t, x) \in [t_0, \vartheta) \times R^n$

$$\inf_{s \in R^n} \sup_{h \in R^n} (\langle s, h\rangle - \partial_- w(t,x)|(1,h) - H(t,x,s)) \geq 0 \tag{1.7}$$

$$\sup_{s \in R^n} \inf_{h \in R^n} (\langle s, h\rangle - \partial_+ w(t,x)|(1,h) - H(t,x,s)) \leqslant 0 \tag{1.8}$$

as well as the boundary condition $w(\vartheta, x) = \sigma(x)$, $x \in R^n$. Here we have used the notation

$$\partial_- w(t,x)|(1,h) = \liminf_{\delta \downarrow 0} \delta^{-1}(w(t+\delta, x+\delta h) - w(t,x))$$

$$\partial_+ w(t,x)|(1,h) = \limsup_{\delta \downarrow 0} \delta^{-1}(w(t+\delta, x+\delta h) - w(t,x))$$

for the lower and upper derivatives, respectively, of w at the point (t, x) in the direction $(1, h)$.

At points where the function is differentiable, inequalities (1.7) and (1.8) become the Hamilton–Jacobi equation (1.4), so they may be regarded as a generalization of that equation. The finite-difference scheme proposed below for the Hamilton–Jacobi equation is convergent and the limit is the utility function, i.e. a function satisfying inequalities (1.7) and (1.8).

Equation (1.4) will be considered in a certain compact domain $G_r \subset T \times R^n$, $r > 0$, defined as follows.

Let $X(t_*, x_*)$ be the set of all solutions $x(t)$ of the differential inclusion

$$\dot{x}(t) \in F(t, x(t)), \quad t \in [t_*, \vartheta], \quad x(t_*) = x_* \tag{1.9}$$

where $F(\tau, y) = \mathrm{co}\{f \in R^n : f = f(\tau, y, u, \upsilon),\ u \in P,\ \upsilon \in Q\}$, $(\tau, y) \in T \times R^n$ is the convex hull of the right-hand side of system (1.1).

Let G be a closed set satisfying the following strong invariance condition

$(G1)$ If $(t_*, x_*) \in G$, then $(t, x(t)) \in G$ for all $x(t) \in X(t_*, x_*)$, $t \in [t_*, \vartheta]$.

By condition $(f3)$ compact domains G exist satisfying this condition.

Let

$$K = \max_{(t,x,u,\upsilon)\in G\times P\times Q} \|f(t,x,u,\upsilon)\|$$

be the maximum velocity of system (1.1) in region G.

Obviously

$$K \le \max_{(t,x)\in G} \kappa(1+\|x\|)$$

Suppose the number r defining the domain G_r is such that $r > K$. Note that for any $(t, x)\in G$ we have $F(t, x)\subset B_r$ (B_r denotes the sphere $\{b\in R^n : \|b\|\leqslant r\}$).

We will define G_r by two conditions:

($G2$) $G_r \subset G$.

($G3$) If $(t_*, x_*)\in G_r$, then $(t, x_* + (t-t_*)B_r)\subset G_r$ for all $t\in[t_*, \vartheta]$.

It follows from conditions ($f1$)–($f4$) and relation (1.6) that the Hamiltonian $H(t, x, s)$: $G_r \times R^n \to R$ satisfies the following conditions:

($H1$) Uniform continuity jointly in all variables.

($H2$) The Lipschitz condition with respect to the variable x

$$|H(t,x_1,s) - H(t,x_2,s)| \le L_1(G_r)\|s\| \, \|x_1 - x_2\|$$

for all $(t, x_1)\in G_r$, $(t, x_2)\in G_r$, $s\in R^n$.

($H3$) The Lipschitz condition with respect to the variable s

$$|H(t,x,s_1) - H(t,x,s_2)| \le K\|s_1 - s_2\| < r\|s_1 - s_2\|$$

for all $(t, x)\in G_r$, $s_1\in R^n$, $s_2\in R^n$.

($H4$) The Lipschitz condition with respect to the variable t

$$|H(t_1,x,s) - H(t_2,x,s)| \le L_2(G_r)\|s\| \, \|t_1 - t_2\|$$

for all $(t_1, x)\in G_r$, $(t_2, x)\in G_r$, $s\in R^n$.

($H5$) Positive homogeneity with respect to the variable s

$$H(t,x,\lambda s) = \lambda H(t,x,s) \text{ for all } (t,x,s)\in G_r\times R^n,\ \lambda\geqslant 0.$$

2. APPROXIMATION OPERATOR FOR HAMILTON–JACOBI EQUATIONS

Let $t\in T$, $t+\Delta\in T$, $t<\vartheta$, $\Delta>0$, $(t, x)\in G_r$. Let us assume that at a time $t+\Delta$ a function $u(\cdot)$ is required which satisfies a Lipschitz condition in the domain $D_{t+\Delta} = \{x\in R^n : (t+\Delta, x)\in G_r, t+\Delta\in T\}$ with constant $L = L(D_{t+\Delta})$. This function will be used in the subsequent constructions as an approximation of the solution $w(t+\Delta, \cdot)$. We define an operator $u\to F(t, \Delta, u)$ approximating the Hamilton–Jacobi equation in the neighbourhood of a point $(t, x)\in G_r$ by a formula that can be interpreted as a generalization of Hopf's formula [9, 11] or of the programmed maximin formula [15, 17] to locally convex hulls

$$\upsilon(x) = F(t,\Delta,u)(x) = f(x) + \sup_{y\in O(x,r\Delta)} \max_{s\in Df(y)} \{\Delta H(t,x,s) + f(y) - f(x) - \langle s, y-x\rangle\} \tag{2.1}$$

Here the function $\upsilon: D_t \to R$ is treated as an approximation of the solution $w(t, \cdot)$ in the domain $D_t = \{s \in R^n : (t, x) \in G_r, t \in T\}$.

The set $O(x, r\Delta)$ is a neighbourhood of x of radius $r\Delta$, $r > K$, $\Delta > 0$, $(t, x) \in G_r$, i.e.

$$O(x, r\Delta) = \{y \in R^n : \|y - x\| < r\Delta\}$$

the function $f(y)$: $\overline{O}(x, r\Delta) \to R$ is the locally convex hull of the function $u(y)$ in the closed neighbourhood $\overline{O}(x, r\Delta)$ of x of radius $r\Delta$

$$f(y) = \inf\left\{\sum_{k=1}^{n+1} \alpha_k u(y_k): \ y_k \in \overline{O}(x, r\Delta), \ \alpha_k \geq 0, \ k = 1, \ldots, n+1\right.$$

$$\left.\sum_{k=1}^{n+1} \alpha_k y_k = y, \ \sum_{k=1}^{n+1} \alpha_k = 1\right\}, \quad y \in \overline{O}(x, r\Delta) \tag{2.2}$$

$$\overline{O}(x, r\Delta) = \{y \in R^n : \|y - x\| \leq r\Delta\}$$

The set $Df(y)$ is the subdifferential [21, 22] of the convex function f at a point y, $y \in O(x, r\Delta)$

$$Df(y) = \{s \in R^n : \ f(z) - f(y) \geq \langle s, z - y\rangle, \ z \in \overline{O}(x, r\Delta)\} \tag{2.3}$$

Note that in formula (2.1)

$$f(y) - f(x) - \langle s, y - x\rangle \leq 0, \quad y \in O(x, r\Delta), \ s \in Df(y)$$

We now consider the properties of locally convex hulls and subdifferentials.†

Lemma 1.

1. The function f: $\overline{O}(x, r\Delta) \to R$ satisfies the following limit

$$|f(z) - f(y)| \leq L\left(1 + \frac{r\Delta + \|y - x\|}{r\Delta - \|y - x\|}\right) \|z - y\|$$

for all $z \in \overline{O}(x, r\Delta)$, $y \in O(x, r\Delta)$. In particular, when $y = x$ the following inequality is true

$$|f(z) - f(x)| \leq 2L\|z - x\|$$

for all $z \in \overline{O}(x, r\Delta)$.

2. The function f: $\overline{O}(x, K\Delta) \to R$, $K < r$ satisfies the Lipschitz condition with constant $L(1 + (r + K)/(r - K))$.

3. For any subgradient $s \in Df(y)$, $y \in O(x, r\Delta)$

$$L(1 + (r + K)/(r - K)).$$

In particular

$$\|s\| \leq L\left(1 + \frac{r\Delta + \|y - x\|}{r\Delta - \|y - x\|}\right)$$

†For the proofs of all these assertions, see TARAS'YEV A. M., Approximation schemes for constructing solutions of the basic equation of the theory of control and differential games. Ekateringburg, 1992. Deposited at VINITI 31.07.92, No. 2543-B92.

$$\|s\| \le 2L, \quad s \in Df(x)$$

$$\|s\| \le L\left(1 + \frac{r+K}{r-K}\right), \quad s \in Df(y), \quad y \in \overline{O}(x, K\Delta)$$

The following example shows that the condition $r > K$ is essential. Let $u = u(y) = u(y_1, y_2) = -|y_2|$, $y = (y_1, y_2) \in R^2$, $x = (0, 0)$, $\Delta = 1$, $r = K = 1$, i.e. $\overline{O}(x, r\Delta) = \overline{O}(x, K\Delta) = \{y \in R^2 : (y_1^2 + y_2^2)^{1/2} \leqslant 1\}$. Clearly, $f(y) = -(1 - y_1^2)^{1/2}$, and so $\partial f / \partial y_1 = y_1(1 - y_1^2)^{-1/2}$. The partial derivative $\partial f / \partial y_1$ increases without limit in absolute value as $|y_1| \to 1$. Consequently, the function $f(y) = -(1 - y_1^2)^{1/2}$ is not Lipschitz continuous in the neighbourhood $\overline{O}(x, K\Delta)$.

Lemma 2. Suppose that the function $\xi(y)$: $\overline{O}(x, r\Delta) \to R$ is convex and Lipschitz continuous, and moreover

$$\xi(y) > \xi(y_0), \quad y \in \overline{O}(x, K\Delta) \setminus \{y_0\}, \quad y_0 \in \partial\overline{O}(x, K\Delta)$$

$$\partial\overline{O}(x, K\Delta) = \{y \in \overline{O}(x, K\Delta): \|y - x\| = K\Delta\}, \quad r > K$$

Then a sequence $\{y_m\}$, $y_m \in O(x, K\Delta)$, $\lim_{m\to\infty} y_m = y_0$, a sequence $\{l_m\}$, $l_m \in D\xi(y_m)$ and a vector $l_0 \in D\xi(y_0) \subset R^n$, $\lim_{m\to\infty} l_m = l_0$ exist such that for all $y \in \overline{O}(x, K\Delta)$

$$\xi(y) - \xi(y_0) \ge \langle l_0, y - y_0 \rangle \ge 0$$

Using Lemmas 1 and 2, one can prove the following properties of the operator F.

Property 1. The value $v(x) = F(t, \Delta, u)(x)$ is well defined for all functions u: $R^n \to R$ which satisfy the Lipschitz condition, where $(t, x) \in G_r$, $t \in T$, $\Delta > 0$, $t + \Delta \in T$. One then has the following limits

$$\min_{y \in \overline{O}(x, r\Delta)} u(y) - 2LK\Delta \le F(t, \Delta, u)(x) \le \max_{y \in \overline{O}(x, K\Delta)} u(y)$$

Property 2. The operator F satisfies the following equalities

$$F(t, \Delta, u)(x) = f(x) + \sup_{y \in O(x, r\Delta)} \max_{s \in Df(y)} \{\Delta H(t, x, s) + f(y) - f(x) - \langle s, y - x \rangle\} =$$

$$= f(x) + \sup_{y \in O(x, K\Delta)} \max_{s \in Df(y)} \{\Delta H(t, x, s) + f(y) - f(x) - \langle s, y - x \rangle\} =$$

$$= f(x) + \max_{y \in \overline{O}(x, K\Delta)} \max_{s \in Df(y)} \{\Delta H(t, x, s) + f(y) - f(x) - \langle s, y - x \rangle\}$$

Thus the supremum in the operator F over the set $O(x, r\Delta)$ is the same as over the set $O(x, K\Delta)$, $r > K$, and is achieved on the set $\overline{O}(x, K\Delta)$.

3. PROPERTIES OF THE OPERATOR F AND GENERAL CONDITIONS FOR THE CONVERGENCE OF APPROXIMATION SCHEMES FOR HAMILTON-JACOBI EQUATIONS

We will now consider some properties of the operator F defined by formula (2.1), which may be related to the sufficient conditions of [4, 13] for the convergence of ASs. The operator F satisfies the series of sufficient conditions presented there. Therefore the explicit AS with the finite-difference operator (2.1) converges, with a convergence error estimate of the order of $\Delta^{1/2}$.

Theorem 1. The operator $u \to F(t, \Delta, u)$ defined by formula (2.1) satisfies the following conditions.

(*F*1) For all $x \in D_t$

$$F(t, 0, u)(x) = u(x)$$

(*F*2) The mapping $(t, \Delta) \to F(t, \Delta, u)$ is continuous. In fact

$$|F(t_1,\Delta_1,u)(x) - F(t_2,\Delta_2,u)(x)| \le 2L(r+K)|\Delta_1 - \Delta_2| +$$
$$+L\left(1+\frac{r+K}{r-K}\right)(K|\Delta_1-\Delta_2| + L_2(G_r)\max\{\Delta_1,\Delta_2\}|t_1-t_2|)$$

(*F*3) For all points $x \in D_t$ and numbers $a \in R$

$$F(t,\Delta,u+a)(x) = F(t,\Delta,u)(x) + a$$

(*F*4) A constant $C_1 \geqslant 0$ exists such that, for all points $x \in D_t$

$$|F(t,\Delta,u)(x) - u(x)| \le C_1$$

where we may put $C_1 = (r+2K)L\Delta$.

(*F*5) If $u_1(x) \geqslant u_2(x)$ for all $x \in D_{t+\Delta}$, then $F(t, \Delta, u_1)(x) \geqslant F(t, \Delta, u_2)(x)$ for all $x \in D_t$.

(*F*6) A constant $C_2 \geqslant 0$ exists such that

$$\|F(t,\Delta,u)\|_{D_t} \le \exp(C_2\Delta)(\|u\|_{D_{t+\Delta}} + C_2\Delta)$$

$$\left(\|F(t,\Delta,u)\|_{D_t} = \max_{x\in D_t}|F(t,\Delta,u)(x)|, \quad \|u\|_{D_{t+\Delta}} = \max_{x\in D_{t+\Delta}}|u(x)|\right)$$

By condition (*H*5) we can put $C_2 = 0$.

(*F*7) A constant C_3 exists such that for all $x_1 \in D_t$, $x_2 \in D_t$

$$|F(t,\Delta,u)(x_1) - F(t,\Delta,u)(x_2)| \le \exp(C_3\Delta)L\|x_1 - x_2\|$$

$$\left(C_3 = L_1(G_r)\left(1+\frac{r+K}{r-K}\right)\right)$$

(*F*8) A parameter C_4 exists such that for all twice differentiable functions $\varphi : D_{t+\Delta} \to R$ and points $x \in D_t \subset D_{t+\Delta}$

$$\left|\frac{F(t,\Delta,\varphi)(x) - \varphi(x)}{\Delta} - H(t,x,\nabla\varphi)(x))\right| \le C_4\Delta$$

$$\left(C_4 = \left(r^2 + 2Kr\left(2+\frac{r+K}{r-K}\right)\right)\|\partial^2\varphi\|\right)$$

where $\nabla\varphi(x)$ is the gradient of φ at a point $x \in D_t$ and $\|\partial^2\varphi\|$ is the norm of the second derivative of φ, i.e.

$$\|\partial^2\varphi\| = \sum_{i,j} \|\partial^2\varphi/\partial x_i\partial x_j\|, \quad \|\partial^2\varphi/\partial x_i\partial x_j\| = \max_{y\in D_{t+\Delta}} |\partial^2\varphi(y)/\partial x_i\partial x_j|$$

$i, j = 1, \dots, n$

The next assertion follows from Theorem 1 and the results of [4, 13], where it was proved that an AS with an operator satisfying conditions ($F1$)–($F8$) is convergent.

Theorem 2. Let w be a generalized solution of problem (1.4), (1.5) in the domain G_r. For a partition $\Gamma = \{t_0 < t_1 < \dots < t_N = \vartheta\}$ of the interval T with constant mesh size $\Delta = t_{i+1} - t_i$ $(i = 0, \dots, N-1)$, define an AS with the operator F of (2.1)

$$\begin{gathered} u_\Gamma(\vartheta, x) = \sigma(x), \quad x \in D_\vartheta \\ u_\Gamma(t, x) = F(t, t_{i+1} - t, u_\Gamma(t_{i+1}, \bullet))(x) \\ t \in [t_i, t_{i+1}), \quad x \in D_t, \quad i = 0, \dots, N-1 \end{gathered} \tag{3.1}$$

Then the AS (3.1) converges to a generalized solution w of problem (1.4), (1.5). Moreover, a constant C exists such that, for sufficiently small Δ,

$$\|u_\Gamma - w\|_{G_r} \le C\Delta^{1/2} \tag{3.2}$$

$$\left(\|u_\Gamma - w\|_{G_r} = \max_{(t,x) \in G_r} |u_\Gamma(t, x) - w(t, x)|\right)$$

4. AN APPROXIMATION OPERATOR ON A GRID

Let us consider the possibility of approximating the operator $F(t, \Delta, u)$ by an operator $F^*(t, \Delta, u)$ whose value is a piecewise-linear function, the vertices of whose graph lie at the points of a fixed grid.

Let $(\tau, x_0) \in G_r$, $h_i = \gamma_i \Delta > 0$ $(i = 1, \dots, n)$. The set of points $\{y = x_0 + \Sigma(m_1 h_1 e_1 + \dots + m_n h_n e_n)\}$ such that $(\tau, y) \in G_r (m_i = 0, \pm 1, \pm 2, \dots, i = 1, \dots, n)$ is called a grid; we shall denote it by $\mathrm{GR}(\tau)$. Here e_i $(i = 1, \dots, n)$ are basis vectors in R^n. Let D_τ^* be the convex hull of the grid $\mathrm{GR}(\tau)$

$$D_\tau^* = \{y \in R^n : \ y = \sum_{j=0}^n \alpha_j y_j, \ \ y_j \in \mathrm{GR}(\tau), \ \ \alpha_j \ge 0, \ \ j = 0, \dots, n, \ \ \sum_{j=0}^n \alpha_j = 1\}$$

We shall assume that $t \in T$, $t + \Delta \in T$ and u: $D_{t+\Delta}^* \to R$ is a function which satisfies the Lipschitz condition. Let Ω be some fixed partition of the n-cube into simplexes.

Define the value of the operator $F^*(t, \Delta, u)(y)$: $D_t^* \to R$ at a point $y \in D_t^*$ by

$$F^*(t, \Delta, u)(y) = \sum_{j=0}^n \alpha_j F(t, \Delta, u)(y_j)$$

$$y \in D_t^*, \ \ y_j \in \mathrm{GR}(t), \ \ \alpha_j \ge 0, \ \ j = 0, \dots, n, \ \ \sum_{j=0}^n \alpha_j = 1, \ \ y = \sum_{j=0}^n \alpha_j y_j$$

$$y_0 = x_0 + \sum (m_1 h_1 e_1 + \dots + m_n h_n e_n)$$

$$y_j = y_0 + \sum (k_1 h_1 e_1 + \dots + k_n h_n e_n), \ \ j = 1, \dots, n, \ \ k_i = 0, \pm 1$$

The coefficients $\alpha_j = \alpha_j(\Omega)$ and points $y_j = y_j(\Omega)$ $(j = 0, \dots, n)$, both here and below, are uniquely defined by the partition Ω.

Theorem 3. The operator $F(t, \Delta, u)$ satisfies conditions ($F1$)–($F8$) with parameters

$$C_1^* = (r + 2K + \sqrt{n} \max_i \{\gamma_i\}) L\Delta, \ \ C_2^* = C_2 = 0$$

$$C_3^* = C_3 = L_1(G_r)\left(1 + \frac{r+K}{r-K}\right)$$

$$C_4^* = C_4 + (n\max_i\{\gamma_i^2\} + \sqrt{n}K\max_i\{\dot\gamma_i\})\|\partial^2\varphi\| + \sqrt{n}L_1(G_r)\max_i\{\gamma_i\}\|\nabla\varphi\|$$

Theorem 4. Let w be a generalized solution of problem (1.4), (1.5) in the domain G_r. For a partition $\Gamma = \{t_0 < t_1 < \ldots < t_N = \vartheta\}$ of the interval T with constant mesh size $\Delta = t_{i+1} - t_i$ $(i = 0, \ldots, N-1)$, define an AS with operator F^*

$$u_\Gamma^*(\vartheta, y) = \sigma^*(y) = \sum_{j=0}^{n} \alpha_j \sigma(y_j), \quad y \in D_\vartheta^*, \quad y = \sum_{j=0}^{n} \alpha_j y_j$$

$$\sum_{j=0}^{n} \alpha_j = 1, \quad \alpha_j = \alpha_j(\Omega) \geq 0, \quad y_j = y_j(\Omega) \in \mathrm{GR}(\vartheta), \quad j = 0, \ldots, n$$

$$u_\Gamma^*(t,x) = F^*(t, t_{i+1} - t, u_\Gamma^*(t_{i+1}, \bullet))(x) \qquad (4.1)$$

$$t \in [t_i, t_{i+1}), \quad x \in D_t^*, \quad i = 0, \ldots, N-1$$

Then the AS (4.1) converges to a generalized solution w of problem (1.4), (1.5). Moreover, a constant C^* exists such that, for sufficiently small Δ

$$\|u_\Gamma^* - w\|_{G_r^*} \leq C^*\Delta^{1/2} \qquad (4.2)$$

where

$$\|u_\Gamma^* - w\|_{G_r^*} = \max_{(t,x)\in G_r^*} |u_\Gamma^*(t,x) - w(t,x)|$$

$$G_r^* = \{(t,x) \in G_r:\ t \in T,\ x \in D_t^*\}$$

$$C^* = 2((C_1^*)^2 + 2L_w L_\sigma + (L_w)^2) + (L_1(G_r) + L_2(G_r))(1 + L_w)(1 + 2(\vartheta - t_0)L_w) +$$
$$+ 2(\vartheta - t_0)C_4^*(1 + 18\overline{R}) + 6(\vartheta - t_0)\overline{R}$$

$$L_\sigma = L_3(D_\vartheta), \quad L_w = L_\sigma \exp(L_1(G_r)(\vartheta - t_0))$$

$$C_1^* \leq (r + 2K)L, \quad L = L_\sigma \exp(C_3^*(\vartheta - t_0))$$

$$\overline{R} = R + 1, \quad R = \|\sigma\|_{D_\vartheta^*} + C_2^*(\vartheta - t_0), \quad \|\sigma\|_{D_\vartheta^*} = \max_{x\in D_\vartheta^*}|\sigma(x)|$$

5. ALGORITHMS FOR COMPUTING VALUES OF THE OPERATOR F

Different types of operators. We will point out some further properties of the operator F. We first introduce some notation. Let

$$F(t, \Delta, r_i, u)(x) = f(r_i, x) + \max_{y\in\overline{O}(x,K\Delta)} \max_{s\in Df(r_i,y)} \{\Delta H(t,x,s) +$$
$$+ f(r_i, y) - f(r_i, x) - \langle s, y - x\rangle\}, \quad i = 1, 2$$

$$F(t, \Delta, S, u)(x) = f(S, x) + \max_{y\in\overline{O}(x,K\Delta)} \max_{s\in Df(S,y)} \{\Delta H(t,x,s) + f(S,y) - f(S,x) - \langle s, y - x\rangle\}$$

where $r_2 > r_1 > K$, the set $S = S(x, r_1, r_2, \Delta)$ is a convex polyhedron such that

$$\overline{O}(x, r_1\Delta) \subset S(x, r_1, r_2, \Delta) \subset \overline{O}(x, r_2\Delta)$$

and the functions $f(r_i, \cdot)$ and $f(S, \cdot)$ are the convex hulls of $u(\cdot)$ over the sets $\overline{O}(x, r_i\Delta)$ $(i = 1, 2)$

and $S(x, r_1, r_2, \Delta)$, respectively.

Property 3.

$$F(t,\Delta,r_2,u)(x) \geq F(t,\Delta,S,u)(x) \geq F(t,\Delta,r_1,u)(x), \quad x \in D_t$$

Remark 1. By property 3, ASs (3.1) and (4.1) with an operator $F = F(t, \Delta, S, u)$ are convergent, since that is the case for ASs (3.1) and (4.1) with operators $F = F(t, \Delta, r_i, u)$ $(i = 1, 2)$. The convergence error estimate is of the order of $\Delta^{1/2}$.

A special feature of this approximation operator F is the appearance in formula (2.1) of a mathematical programming problem. If the function f in (2.1) is piecewise-linear and the Hamiltonian $H(t, x, s)$ is piecewise-linear and positively homogeneous as a function of the impulse variable s, the mathematical programming problem may be reduced to the solution of a series of linear programming problems.

Indeed, suppose that u is piecewise-linear. Then the convex hull $f(\cdot) = f(S, \cdot)$ of u over the convex polyhedron $S(x, r_1, r_2, \Delta)$ is also piecewise-linear. In particular, in the neighbourhood $O(x, K\Delta)$ we can write f as

$$f(y) = \max_j \max_n (\langle l_n^j, y - y_j \rangle + f(y_j)), \quad j = 1,\ldots,N_g, \quad n = 1,\ldots,N_j$$

where the points y_j and the vectors l_n^j satisfy the following condition: an $i_0 \in J(y)$, exists such that for all $i \in J(y)$

$$\operatorname{co} L(y,i) \subseteq \operatorname{co} L(y,i_0), \quad y \in O(x,K\Delta)$$

$$J(y) = \{i\colon \max_j \max_n (\langle l_n^j, y - y_j \rangle + f(y_j)) = \max_n (\langle l_n^i, y - y_i \rangle + f(y_i))\}$$

$$L(y,i) = \{l = l_k^i\colon \max_n (\langle l_n^i, y - y_i \rangle + f(y_i)) = \langle l_k^i, y - y_i \rangle + f(y_i)\}, \quad i \in J(y)$$

The subdifferential $Df(y)$ of f at $y \in O(x, K\Delta)$ is defined by

$$Df(y) = \operatorname{co} L(y,i_0)$$

If the relation $i_0 = i_0(y_j) = j$ holds at the points y_j, $j = 1, \ldots, N_g$, then the subdifferential $Df(y_j)$ of f at the point $y_j \in O(x, K\Delta)$ is the convex polyhedron defined by the formula

$$Df(y_j) = \operatorname{co}\{l_n^j, \ n = 1,\ldots,N_j\}, \quad j = 1,\ldots,N_g$$

We may assume without loss of generality that the Hamiltonian $H(t, x, s)$ is piecewise-linear and positively homogeneous as a function of s. In particular, $H(t, x, s)$ will satisfy these conditions if the control system is linear in the control variables and the constraints on the controls are polyhedra. In that case

$$H(t,x,s) = \langle s, h(t,x) \rangle + \min_{u \in P} \langle s, B(t,x)u \rangle + \max_{\upsilon \in Q} \langle s, C(t,x)\upsilon \rangle$$

where P and Q are convex polyhedra.

Let u_k be the vertices of the polyhedron $B(t, x)P$ and let L_k^u be the cones of linearity of the function $s \to \min_{u \in P} \langle s, B(t, x)u \rangle$, i.e.

$$L_k^u = L_k^u(t,x) = \{s \in R^n\colon \langle s, u - u_k \rangle \geq 0, \ u \in B(t,x)P\}, \quad k = 1,\ldots,N_u$$

Similarly, let υ_m be the vertices of the polyhedron $C(t, x)Q$ and L_m^υ the cones of linearity of the function $s \to \max_{\upsilon \in Q}\langle s, C(t, x)\upsilon\rangle$, i.e.

$$L_m^{\upsilon} = L_m^{\upsilon}(t,x) = \{s \in R^n : \langle s, \upsilon - \upsilon_m\rangle \le 0, \ \ \upsilon \in C(t,x)Q\}, \ \ m = 1,\dots,N_{\upsilon}$$

Property 4. If u is piecewise-linear and the Hamiltonian H is piecewise-linear and positively homogeneous as a function of s, then F may be computed from the formula

$$F = F(t,\Delta,S,u)(x) = f(x) + \max_j \max_k \max_m \max_s \{\Delta(\langle s, h(t,x)\rangle + \langle s, u_k\rangle +$$

$$+\langle s, \upsilon_m\rangle) + f(y_j) - f(x) - \langle s, y_j - s\rangle\} \tag{5.1}$$

$$s \in L_{j,k,m}(t,x) = Df(y_j) \cap L_k^u \cap L_m^{\upsilon}$$

In this formula the set $L_{j,k,m}(t, x)$ is a convex polyhedron, while the function being maximized is linear in s. Thus, calculation of the value $F(t, \Delta, u)(x)$ of the operator F at a point x reduces to a series of linear programming problems.

Remark 2. Consider the finite-difference operator G dual to the F of (2.1)

$$G(t,\Delta,u)(x) = g(x) + \inf_{y \in O(x,r\Delta)} \min_{s \in \overline{D}g(y)} \{\Delta H(t,x,s) + g(y) - g(x) - \langle s, y - x\rangle\} \tag{5.2}$$

$$t \in T, \ \ t+\Delta \in T, \ \ t < \vartheta, \ \ \Delta > 0, \ \ (t,x) \in G_r, \ \ r > K$$

where $g(y)$: $\overline{O}(x, r\Delta) \to R$ is the locally concave hull of $u(y)$ in the closed neighbourhood $\overline{O}(x, r\Delta)$ of x of radius $r\Delta$

$$g(y) = \sup\left\{\sum_{k=1}^{n+1} \alpha_k u(y_k) : \ y_k \in \overline{O}(x,r\Delta), \ \alpha_k \ge 0, \ k = 1,\dots,n+1\right.$$

$$\left.\sum_{k=1}^{n+1} \alpha_k y_k = y, \ \sum_{k=1}^{n+1} \alpha_k = 1\right\}, \ \ y \in \overline{O}(x,r\Delta)$$

and the set $\overline{D}g(y)$ is the superdifferential of the concave function g at a point y, $y \in O(x, r\Delta)$

$$\overline{D}g(y) = \{s \in R^n : \ g(z) - g(y) \le \langle s, z - y\rangle, \ \ z \in \overline{O}(x,r\Delta)\}$$

Remark 3. It can be shown that G satisfies conditions (F1)–(F8). Consequently, the AS (3.1) and (4.1) with operator G is convergent, with convergence error estimate of the order of $\Delta^{1/2}$.

Remark 4. The operator G possesses properties 1–4.

Remark 5. The following inequality holds

$$G(t,\Delta,u)(x) \ge F(t,\Delta,u)(x), \ \ x \in D_t$$

Remark 6. Let $\alpha_1(x)$, $\alpha_2(x)$ be such that $\alpha_i(x) \geqslant (0)$ $(i = 1, 2)$ and $\alpha_1(x) + \alpha_2(x) = 1$. Then the operator

$$E(t,\Delta,u)(x) = \alpha_1(x)F(t,\Delta,u)(x) + \alpha_2(x)G(t,\Delta,u)(x)$$

satisfies the inequalities

$$F(t,\Delta,u)(x) \le E(t,\Delta,u)(x) \le G(t,\Delta,u)(x), \ \ x \in D_t \tag{5.3}$$

The AS (3.1) and (4.1) with the operator E converges, with error estimate of the order of $\Delta^{1/2}$, for any (not necessarily continuous) weighting functions $x \to \alpha_1(x)$, $x \to a_2(x)$.

Remark 7. Let γ_i be numbers such that

$$K < \left(\sum_{i=1}^{n} \gamma_i^{-2} \right)^{-1/2}, \quad i = 1,\ldots,n \tag{5.4}$$

(in particular, if $\gamma_i = \gamma$ $(i = 1, \ldots, n)$ this means that $K\sqrt{(n < y)})$. Define

$$r_1 = \left(\sum_{i=1}^{n} \gamma_i^{-2} \right)^{-1/2}, \quad r_2 = \max_i \gamma_i, \quad i = 1,\ldots,n$$

$$S(x, r_1, r_2, \Delta) = \mathrm{co}\{x \pm \Delta\gamma_i e_i, \ i = 1,\ldots,n\}$$

$$u(y) = \sum_{j=0}^{n} \alpha_j u(y_j), \quad y \in S(x, r_1, r_2, \Delta) \tag{5.5}$$

$$y_0 = x, \quad y_i = x \pm \Delta\gamma_i e_i, \quad i = 1,\ldots,n$$

$$\alpha_j \geq 0, \quad j = 0,\ldots,n, \quad \sum_{j=0}^{n} \alpha_j = 1, \quad y = \sum_{j=0}^{n} \alpha_j y_j$$

The function f is then piecewise-linear and defined in the elementary "rhombus" $S = S(x, r_1, r_2, \Delta)$ by the relations

$$f(x \pm \Delta\gamma_i e_i) = u(x \pm \Delta\gamma_i e_i), \quad i = 1,\ldots,n$$

$$f(x) = \min\{u(x), \ \min_i \{1/2(u(x + \Delta\gamma_i e_i) + u(x - \Delta\gamma_i e_i))\}\}$$

$$f(y) = \sum_{j=0}^{n} \alpha_j f(y_j), \quad y \in S(x, r_1, r_2, \Delta)$$

$$y_0 = x, \quad y_i = x \pm \Delta\gamma_i e_i, \quad i = 1,\ldots,n$$

$$\alpha_j \geq 0, \quad j = 0,\ldots,n, \quad \sum_{j=0}^{n} \alpha_j = 1, \quad y = \sum_{j=0}^{n} \alpha_j y_j$$

The subdifferential $Df(x)$ of f at the point x is a rectangular parallelepiped with faces parallel to the coordinate axes

$$Df(x) = \mathrm{co}\{a_k : \ k = 1,\ldots,2^n\}$$

$$a_k = (a_k^1,\ldots,a_k^n)$$

$$a_k^i = \pm(f(x \pm \Delta\gamma_i e_i) - f(x))(\Delta\gamma_i)^{-1}, \quad i = 1,\ldots,n$$

The operator F is computed by the formula

$$F = F(t, \Delta, S, u)(x) = f(x) + \Delta \max_{s \in Df(x)} H(t, x, s) =$$

$$= f(x) + \Delta \max_k \max_m \max_s \{\langle s, h(t,x)\rangle + \langle s, u_k\rangle + \langle s, \upsilon_m\rangle\} \tag{5.6}$$

$$s \in L_{k,m}(t,x) = Df(x) \cap L_k^u \cap L_m^\upsilon, \quad L_k^u = L_k^u(t,x), \quad L_m^\upsilon = L_m^\upsilon(t,x)$$

Remark 8. Assume that condition (5.4) holds. Then

$$G = G(t,\Delta,S,u)(x) = g(x) + \Delta \min_{s \in \overline{D}g(x)} H(t,x,s) =$$
$$= g(x) + \Delta \min_k \min_m \min_s \{\langle s, h(t,x)\rangle + \langle s, u_k\rangle + \langle s, \upsilon_m\rangle\} \tag{5.7}$$
$$s \in L_{k,m}(t,x) = \overline{D}g(x) \cap L_k^u \cap L_m^\upsilon, \quad L_k^u = L_k^u(t,x), \quad L_m^\upsilon = L_m^\upsilon(t,x)$$

$$g(x \pm \Delta\gamma_i e_i) = u(x \pm \Delta\gamma_i e_i), \quad i = 1,\ldots,n$$
$$g(x) = \max\{u(x), \ \max_i\{\tfrac{1}{2}(u(x+\Delta\gamma_i e_i) + u(x-\Delta\gamma_i e_i))\}\}$$
$$\overline{D}g(x) = \mathrm{co}\{b_k : \ k = 1,\ldots,2^n\}$$
$$b_k = (b_k^1,\ldots,b_k^n)$$
$$b_k^i = \pm(g(x \pm \Delta\gamma_i e_i) - g(x))(\Delta\gamma_i)^{-1}, \quad i = 1,\ldots,n$$

Remark 9. Assume that condition (5.4) holds and let $f(x) < g(x)$ in (5.6), (5.7). Define

$$\alpha_1(x) = \frac{g(x)-u(x)}{g(x)-f(x)}, \quad \alpha_2(x) = \frac{u(x)-f(x)}{g(x)-f(x)} \tag{5.8}$$

In that case

$$E = E(t,\Delta,S,u)(x) = \alpha_1(x)F(t,\Delta,S,u)(x) + \alpha_2(x)G(t,\Delta,S,u)(x) =$$
$$= u(x) + \Delta(\alpha_1(x) \max_{s \in Df(x)} H(t,x,s) + \alpha_2(x) \min_{s \in \overline{D}g(x)} H(t,x,s)) =$$
$$= u(x) + \Delta\left(\max_{s \in D_* u(x)} H(t,x,s) + \min_{s \in D^* u(x)} H(t,x,s)\right) \tag{5.9}$$
$$D_* u(x) = \alpha_1(x) Df(x), \quad D^* u(x) = \alpha_2(x) \overline{D}g(x)$$

It can be shown that the set $D_* u(x)$ is the subdifferential and $D^* u(x)$ is the superdifferential of the function u of (5.5) at x in the sense of Dem'yanov [20], i.e.

$$u(x+h) - u(x) = \partial u(x)|(h) = \lim_{\delta \downarrow 0} \delta^{-1}(u(x+\delta h) - u(x)) = \max_{s \in D_* u(x)} \langle s,h\rangle + \min_{s \in D^* u(x)} \langle s,h\rangle$$

Remark 10. The following relations are satisfied

$$Df(x) \cap \overline{D}g(x) \neq \emptyset, \quad c \in Df(x) \cap \overline{D}g(x)$$
$$c = (c^1,\ldots,c^n) \tag{5.10}$$
$$c^i = (u(x+\Delta\gamma_i e_i) - u(x-\Delta\gamma_i e_i))(2\Delta\gamma_i)^{-1}, \quad i = 1,\ldots,n$$
$$\max_{s \in Df(x)} H(t,x,s) \geq \min_{s \in \overline{D}g(x)} H(t,x,s)$$

Remark 11. The operator E is defined by the equalities

$$E = u(x) + \Delta H(t,x,c), \quad f(x) = g(x) = u(x) \tag{5.11}$$

$$E = u(x) + \Delta \max_{s \in Du(x)} H(t,x,s), \quad f(x) = u(x) < g(x) \tag{5.12}$$

$$E = u(x) + \Delta \min_{s \in \overline{D}u(x)} H(t,x,s), \quad f(x) < u(x) = g(x) \tag{5.13}$$

Formulae (5.12) and (5.13) may be interpreted as a Godunov operator or as Hopf's formula in the Riemann problem for a convex and concave boundary function [5, 9, 11, 12].

Remark 12. Let us write (5.9) in the form

$$E = u(x) + \Delta(\alpha_1(x) \max_{s \in Df(x)} H(t,x,s) + \alpha_2(x) \min_{s \in \bar{D}g(x)} H(t,x,s)) =$$
$$= u(x) - \beta(u(x) - f(x)) + \Delta \max_{s \in Df(x)} H(t,x,s) \tag{5.14}$$

$$\beta = \frac{\Delta(M-m)}{g(x)-f(x)}, \quad 0 \le \beta \le 1$$
$$M = \max_{s \in Df(x)} H(t,x,s), \quad m = \min_{s \in \bar{D}g(x)} H(t,x,s)$$

Taking (5.14) into account, we can write the operator E as the operator F on the set $S_\beta = \mathrm{co}\{x \pm \beta\Delta\gamma_i e_i,\ i = 1, \ldots, n\}$

$$S_\beta = \mathrm{co}\{x \pm \beta\Delta\gamma_i e_i,\ i = 1,\ldots,n\}$$
$$E = E(t,\Delta,S,u)(x) = F(t,\Delta,S_\beta,u)(x) \tag{5.15}$$

On the other hand, E may be written in the form

$$E(t,\Delta,S,u)(x) = u(x) + \beta(g(x) - u(x)) + \Delta \min_{s \in \bar{D}g(x)} H(t,x,s) = G(t,\Delta,S_\beta,u)(x) \tag{5.16}$$

We see from (5.15) and (5.16) that the operators F and G are identical on the set S_β. This indicates that the operator (5.9) yields the exact (or nearly exact) value of the solution (the utility function) at the point x for a problem with simple Hamiltonian (simple motion) and positively homogeneous (not necessarily convex or concave) payoff function. This problem is known in the theory of first-order partial differential equations as the Riemann problem [11].

6. COMPARISON OF APPROXIMATION OPERATORS

We shall now demonstrate the relationship between the operators F (5.6), G (5.7), E (5.9) and approximation operators known from the theory of partial differential equations, such as the Godunov and Lax–Friedrichs operators.

The formula for the Lax–Friedrichs approximation operator may be written as follows [10, 12]:

$$\mathrm{L}F(t,\Delta,u)(x) = \left(1 - \sum_{i=1}^{n} \alpha_i\right) u(x) + \tfrac{1}{2} \sum_{i=1}^{n} \alpha_i (u(x + \Delta\gamma_i e_i) + u(x - \Delta\gamma_i e_i)) + \Delta H(t,x,c) \tag{6.1}$$

(where the vector c is defined by (5.10)).

The operators F, G and LF, with $\alpha_i = 2/(2n+1)$ $(i = 1, \ldots, n)$ and $\gamma > nK$, satisfy the inequalities

$$F(t,\Delta,u)(x) \le \mathrm{LF}(t,\Delta,u)(x) \le G(t,\Delta,u)(x), \quad x \in D_t^*$$

If $f(x) = g(x)$, then

$$F(t,\Delta,u)(x) = \mathrm{LF}(t,\Delta,u)(x) = G(t,\Delta,u)(x) = E(t,\Delta,u)(x)$$

Now consider Godunov's approximation operator [5, 12]

$$\mathrm{GOD}(t,\Delta,u)(x) = u(x) + \Delta \operatorname*{ext}_{s_1 \in I(s_1^-, s_1^+)} \cdots \operatorname*{ext}_{s_n \in I(s_n^-, s_n^+)} H(t,x,s_1,\ldots,s_n) \tag{6.2}$$

$$s_i^+ = (u(x + \Delta\gamma_i e_i) - u(x))(\Delta\gamma_i)^{-1}$$
$$s_i^- = -(u(x - \Delta\gamma_i e_i) - u(x))(\Delta\gamma_i)^{-1}$$

$$I(a,b) = [\min(a,b), \max(a,b)]$$

$$\operatorname*{ext}_{s\in I(a,b)} = \begin{cases} \min\limits_{0\le s\le b}, & a \le b \\ \max\limits_{b\le s\le a}, & a > b \end{cases}$$

The operators F, G and GOD satisfy the relations

$$F(t,\Delta,u)(x) \le \mathrm{GOD}(t,\Delta,u)(x) \le G(t,\Delta,u)(x), \quad x \in D_t^*$$

$$F(t,\Delta,u)(x) = E(t,\Delta,u) = \mathrm{GOD}(t,\Delta,u)(x), \quad f(x) = u(x)$$

$$G(t,\Delta,u)(x) = E(t,\Delta,u)(x) = \mathrm{GOD}(t,\Delta,u)(x), \quad g(x) = u(x)$$

The work reported in this paper was carried out with the financial support of the Russian Fund for Fundamental Research (93-011-16032).

REFERENCES

1. SUBBOTIN A. I., A generalization of the fundamental equation of the theory of differential games. *Dokl. Akad. Nauk SSSR* **254**, 2, 293–297, 1980.
2. SUBBOTIN A. I., *Minimax Inequalities and Hamilton–Jacobi Equations*. Nauka, Moscow, 1991.
3. CRANDALL M. G. and LIONS P.-L., Viscosity solutions of Hamilton–Jacobi equations. *Trans. Am. Math. Soc.* **277**, 1, 1–42, 1983.
4. CRANDALL M. G. and LIONS P.-L., Two approximations of solutions of Hamilton–Jacobi equations. *Math. Comput.* **43**, 167, 1–19, 1984.
5. GODUNOV S K., A difference method for the numerical computation of discontinuous solutions of the equations of hydrodynamics. *Mat. Sbornik* **47**, 3, 271–306, 1959.
6. KRUZHKOV S. N., Methods of constructing generalized solutions of the Cauchy problem for a first-order quasilinear equation. *Uspekhi Mat. Nauk* **20**, 6, 112–118, 1965.
7. OLEINIK O. A., The construction of a generalized solution of the Cauchy problem for a first-order quasilinear equation by introducing a "truncating viscosity". *Uspekhi Mat. Nauk* **14**, 2, 159–164, 1959.
8. FLEMING W. H., The convergence problem for differential games. *J. Math. Anal. Appl.* **3**, 1, 102–116, 1961.
9. HOPF E., Generalized solutions of non-linear equations of first order. *J. Math. Mech.* **14**, 6, 951–973, 1965.
10. LAX P. D., Weak solutions of nonlinear hyperbolic equations and their numerical computation. *Comm. Pure Appl. Math.* **7**, 1, 159–193, 1954.
11. BARDI M. and OSHER S., The nonconvex multi-dimensional Riemann problem for Hamilton–Jacobi equations. *SIAM J. Math. Anal.* **22**, 2, 344–351, 1991.
12. OSHER S. and SHU, C.-W., High order essentially nonoscillatory schemes for Hamilton–Jacobi equations. *SIAM J. Numer. Anal.* **28**, 4, 907–922, 1991.
13. SOUGANIDIS P. E., Approximation schemes for viscosity solutions of Hamilton–Jacobi equations. *J. Diff. Eq.* **59**, 1, 1–43, 1985.
14. KRASOVSKII N. N., The problem of unification of differential games. *Dokl. Akad. Nauk SSSR* **226**, 6, 1260–1263, 1976.
15. KRASOVSKII N. N. and SUBBOTIN A. I., *Positional Differential Games*. Nauka, Moscow, 1974.
16. SUBBOTIN A. I. and TARAS'YEV A. M., Conjugate derivatives of the utility function of a differential game. *Dokl. Akad. Nauk SSSR* **283**, 3, 559–564, 1985.
17. USHAKOV V. N., On the problem of constructing stable bridges in a differential pursuit-escape game. *Izv. Akad. Nauk SSSR. Tekhn. Kibern.* 4, 29–36, 1980.
18. GUSEINOV H. G., SUBBOTIN A. I. and USHAKOV V. N., Derivatives for multivalued mappings with applications to game-theoretical problems of control. *Probl. Contr. Inform. Theory* **14**, 3, 155–167, 1985.
19. TARAS'YEV A. M., USPENSKII A. A. and USHAKOV V. N., The construction of solving procedures in a linear control problem. In *The Lyapunov Functions Method and Applications*, pp. 111–115. Balzer, Basel, 1990.
20. DEM'YANOV V. F., *Minimax: Directional Differentiability*. Izv. Leningrad. Gos. Univ., Leningrad, 1974.
21. CLARKE F. H., *Optimization and Nonsmooth Analysis*. John Wiley, New York, 1983.
22. ROCKAFELLAR R. T., *Convex Analysis*. Princeton University Press, Princeton, NJ, 1970.

Translated by D.L.